Cómo construir familias biespectrales de polinomios ortogonales a partir de familias clásicas[1]

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[1] trabajo conjunto con Antonio J. Durán

Outline

1. **Introduction**
   - Classical orthogonal polynomials
   - Krall orthogonal polynomials
2. **Methodology**
   - $\mathcal{D}$-operators
   - Choice of arbitrary polynomials
   - Identifying the measure
3. **Examples**
   - Charlier, Meixner and Krawtchouk polynomials
   - Laguerre polynomials

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The space $L^2_\omega(\mathcal{S})$

Let ω be a positive measure on $\mathcal{S} \subset \mathbb{R}$ and consider the space of functions $L^2_\omega(\mathcal{S})$ with the inner product

$$\langle f, g \rangle_\omega = \int_\mathcal{S} f(x)g(x)d\omega(x)$$

We say that $f \in L^2_\omega(\mathcal{S})$ if $\langle f, f \rangle_\omega = \|f\|^2_\omega < \infty$.

$\mathcal{S}$ can be a continuous interval, a discrete set of points or a combination of both. The discrete component of the measure is usually written as

$$\omega_d(x) = \sum_{x=0}^{N} a_x \delta_{t_x}, \quad t_{x_0}, \ldots, t_{x_N} \in \mathbb{R}$$

In that case the inner product can be thought of as

$$\langle f, g \rangle_{\omega_d} = \sum_{x=0}^{N} a_x \int f(x)g(x)\delta_{t_x} = \sum_{x=0}^{N} a_x f(t_x)g(t_x)$$

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Orthogonal polynomials

A system of polynomials $(p_n)_n = \{p_0(x), p_1(x), \dots\}$ with $\deg(p_n) = n$ is orthogonal in $L^2_\omega(\mathcal{S})$ if (Gramm-Schmidt)

$$\langle p_n, p_m \rangle_\omega = \int_{\mathcal{S}} p_n(x) p_m(x) d\omega(x) = \|p_n\|^2_\omega \delta_{nm}, \quad n, m \geq 0$$

Every family of OP's $(p_n)_n$ satisfy a three-term recurrence relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad n \geq 1$$

where $a_n, c_n \neq 0$, $b_n \in \mathbb{R}$ and $p_0(x) = 1, p_{-1}(x) = 0$.
Jacobi operator (tridiagonal):

$$Jp = \begin{pmatrix} b_0 & a_1 & & & \\ c_1 & b_1 & a_2 & & \\ & c_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = xp, \quad x \in \mathcal{S}$$

The converse result is also true (Favard's or spectral theorem)

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Classical families (continuous case)

Bochner problem, 1929

$$\sigma(x)\frac{d^2}{dx^2}p_n(x)+\tau(x)\frac{d}{dx}p_n(x)+\lambda_n p_n(x)=0,\quad x\in\mathcal{S}\subset\mathbb{R}$$

$\deg\sigma\le 2,\quad \deg\tau=1$

- Hermite (Normal, Gaussian): $\omega(x)=e^{-x^2}$, $x\in\mathbb{R}$

$$H_n(x)''-2xH_n(x)'=-2nH_n(x)$$

- Laguerre (Gamma, Exponential): $\omega(x)=x^{\alpha}e^{-x}$, $x>0$, $\alpha>-1$

$$xL_n^{\alpha}(x)''+(\alpha+1-x)L_n^{\alpha}(x)'=-nL_n^{\alpha}(x)$$

- Jacobi (Beta, Uniform): $\omega(x)=x^{\alpha}(1-x)^{\beta}$, $x\in(0,1)$, $\alpha,\beta>-1$

$$x(1-x)P_n^{(\alpha,\beta)}(x)''+(\alpha+1-(\alpha+\beta+2)x)P_n^{(\alpha,\beta)}(x)'=$$
$$-n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)$$

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Classical families (discrete case)

If we set

$$\Delta f(x) = f(x+1) - f(x), \quad \nabla f(x) = f(x) - f(x-1)$$

the classification problem is to find discrete OP's $(p_n)_n$

$$\sigma(x)\Delta\nabla p_n(x) + \tau(x)\Delta p_n(x) + \lambda_n p_n(x) = 0, \quad x \in \mathcal{S} \subset \mathbb{N}$$
$$\deg\sigma \leq 2, \quad \deg\tau = 1$$

In other words, if we call the shift operator

$$\mathfrak{s}_j f(x) = f(x+j)$$

the difference equation reads

$$[\sigma(x) + \tau(x)]\mathfrak{s}_1 p_n(x) - [2\sigma(x) + \tau(x)]\mathfrak{s}_0 p_n(x)$$
$$+ \sigma(x)\mathfrak{s}_{-1} p_n(x) + \lambda_n p_n(x) = 0, \quad x \in \mathcal{S} \subset \mathbb{N}$$

Classical families (discrete case)

- Charlier (Poisson):

$$\omega_a(x) = \sum_{x=0}^{\infty} \frac{a^x}{x!}\delta_x, \quad a > 0$$

$$a c_n^a(x+1) - (x+a) c_n^a(x) + x c_n^a(x-1) = -n c_n^a(x)$$

- Meixner (Pascal, Geometric):

$$\omega_{a,c}(x) = \sum_{x=0}^{\infty} \frac{(c)_x a^x}{x!}\delta_x, \quad 0 < a < 1, \quad c > 0$$

$(i)_j = i(i+1)\cdots(i+j-1)$ is the Pochhammer symbol

$$a(x+c) m_n^{a,c}(x+1) - (x + a(x+c)) m_n^{a,c}(x) + x m_n^{a,c}(x-1) = n(a-1) m_n^{a,c}(x)$$

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- Krawtchuok (Binomial, Bernoulli):

$$\omega_{p,N}(x) = \sum_{x=0}^{N} \binom{N}{x} p^x (1-p)^{N-x} \delta_x, \quad 0 < p < 1$$

$$p(N-x)k_n^{p,N}(x+1) - [p(N-x) + x(1-p)]k_n^{p,N}(x) + x(1-p)k_n^{p,N}(x-1) = -nk_n^{p,N}(x)$$

- Hahn (Hypergeometric):

$$\omega_{\alpha,\beta,N}(x) = \sum_{x=0}^{N} \binom{\alpha+x}{x} \binom{\beta+N-x}{N-x} \delta_x, \quad \alpha, \beta > -1, \alpha, \beta < -N$$

$$B(x)Q_n^{\alpha,\beta,N}(x+1) - [B(x) + D(x)]Q_n^{\alpha,\beta,N}(x) + D(x)Q_n^{\alpha,\beta,N}(x-1) = n(n+\alpha+\beta+1)Q_n^{\alpha,\beta,N}(x)$$

where $B(x) = (x+\alpha+1)(x-N)$ and $D(x) = x(x-\beta-N-1)$.

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Krall polynomials (continuous case)

GOAL (Krall, 1939): find families of OP's $(q_n)_n$ which are also eigenfunctions of a higher-order differential operator of the form

$$D_c = \sum_{j=0}^{2m} h_j(x)\frac{d^j}{dx^j}, \quad \deg(h_j) \leq j \quad \Rightarrow \quad D_c(q_n) = \lambda_n q_n$$

Littlejohn, Grünbaum, Heine, Iliev, Koekoek's, Lesky, Bavinck, van Haeringen, Horozov, Koornwinder, etc (80's, 90's, 00's).
Common techniques: ad-conditions, Darboux process, etc.

$(q_n)_n$ are typically orthogonal with respect to the measure

$$\omega(x) + \sum_{j=0}^{m-1} a_j \delta_{x_0}^{(j)}, \quad a_j \in \mathbb{R}$$

where ω is a (modified) classical weight and x_0 is an endpoint of the support of orthogonality of ω.

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The same question arise in the discrete setting, i.e. find families of OP's $(q_n)_n$ which are also eigenfunctions of a higher order difference operator

$$D_d = \sum_{j=-s}^{s} h_j(x)\mathfrak{S}_j, \quad h_s, h_{-s} \neq 0, \quad \Rightarrow \quad D_d(q_n) = \lambda_n q_n$$

The same techniques of adding deltas does not work for the discrete case.

Surprisingly, it has not been until very recently (Durán, 2012) when the first examples appeared ($\mathcal{D}$-operators).

$(q_n)_n$ are typically orthogonal with respect to the measure

$$\omega^F(x) = \prod_{f\in F}(x-f)\,\omega(x)$$

where ω is a discrete classical weight and F is a finite set of numbers. This is also called a Christoffel transform of ω.

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$\mathcal{D}$-OPERATORS

Let $\mathcal{A}$ be an algebra of (differential or difference) operators and $(p_n)_n$ a family of polynomials such that there exists $D_p \in \mathcal{A}$ with $D_p(p_n) = np_n$. Given a sequence of numbers $(\varepsilon_n)_n$, let us consider the operator

$$\mathcal{D}(p_n) = \sum_{j=1}^{n} (-1)^{j+1} \varepsilon_n \cdots \varepsilon_{n-j} p_{n-j} = \varepsilon_n p_{n-1} - \varepsilon_n \varepsilon_{n-1} p_{n-2} + \cdots$$

We say that $\mathcal{D}$ is an $\mathcal{D}$-operator associated with $\mathcal{A}$ and $(p_n)_n$ if $\mathcal{D} \in \mathcal{A}$.

- Laguerre: $\varepsilon_n = -1 \Rightarrow \mathcal{D} = \frac{d}{dx}$.
- Charlier: $\varepsilon_n = 1 \Rightarrow \mathcal{D} = \nabla$.
- Meixner:
  $\varepsilon_n^1 = \frac{a}{1-a} \Rightarrow \mathcal{D}_1 = \frac{a}{1-a}\Delta, \quad \varepsilon_n^2 = \frac{1}{1-a} \Rightarrow \mathcal{D}_2 = \frac{1}{1-a}\nabla.$
- Krawtchouk:
  $\varepsilon_n^1 = \frac{1}{1-a} \Rightarrow \mathcal{D}_1 = \frac{1}{1-a}\nabla, \quad \varepsilon_n^2 = -\frac{a}{1-a} \Rightarrow \mathcal{D}_2 = -\frac{a}{1-a}\Delta.$

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  $\varepsilon_n^1 = \frac{a}{1-a} \Rightarrow \mathcal{D}_1 = \frac{a}{1-a}\Delta, \quad \varepsilon_n^2 = \frac{1}{1-a} \Rightarrow \mathcal{D}_2 = \frac{1}{1-a}\nabla.$
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$\mathcal{D}$-OPERATORS

Let $\mathcal{A}$ be an algebra of (differential or difference) operators and $(p_n)_n$ a family of polynomials such that there exists $D_p \in \mathcal{A}$ with $D_p(p_n) = np_n$. Given a sequence of numbers $(\varepsilon_n)_n$, let us consider the operator

$$\mathcal{D}(p_n) = \sum_{j=1}^{n} (-1)^{j+1} \varepsilon_n \cdots \varepsilon_{n-j} p_{n-j} = \varepsilon_n p_{n-1} - \varepsilon_n \varepsilon_{n-1} p_{n-2} + \cdots$$

We say that $\mathcal{D}$ is an $\mathcal{D}$-operator associated with $\mathcal{A}$ and $(p_n)_n$ if $\mathcal{D} \in \mathcal{A}$.

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$\mathcal{D}$-OPERATORS

THEOREM (DURÁN, 2013)

Let $\mathcal{A}$, $(p_n)_n$, $D_p(p_n) = np_n$, $(\varepsilon_n)_n$ and $\mathcal{D}$.
For an arbitrary polynomial R such that $R(n) \neq 0$, $n \geq 0$, we define a new polynomial P by

$$P(x) - P(x-1) = R(x)$$

and a sequence of polynomials $(q_n)_n$ by $q_0 = 1$ and

$$q_n = p_n + \beta_n p_{n-1}, \quad n \geq 1$$

where the numbers β_n, $n \geq 0$, are given by

$$\beta_n = \varepsilon_n \frac{R(n)}{R(n-1)}, \quad n \geq 1$$

Then there exist $D_q \in \mathcal{A}$ sucht that $D_q(q_n) = P(n)q_n$ where

$$D_q = P(D_p) + \mathcal{D}R(D_p)$$

$\mathcal{D}$-OPERATORS

GOAL: Extend the previous Theorem for the case that we consider a linear combination of $m+1$ consecutive p_n's:

$$q_n = p_n + \beta_{n,1} p_{n-1} + \beta_{n,2} p_{n-2} + \cdots + \beta_{n,m} p_{n-m}$$

Let $R_1, R_2, \ldots, R_m$ be m arbitrary polynomials and m $\mathcal{D}$-operators $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m$ defined by the sequences $(\varepsilon_n^h)_n$, $h = 1, \ldots, m$.

Define the auxiliary functions $\xi_{n,i}^h$ by

$$\xi_{n,i}^h = \varepsilon_n^h \varepsilon_{n-1}^h \cdots \varepsilon_{n-i+1}^h$$

and assume that the following Casorati determinant never vanish ($n \geq 0$)

$$\Omega(n) = \begin{vmatrix} \xi_{n-1,m-1}^1 R_1(n-1) & \xi_{n-2,m-2}^1 R_1(n-2) & \cdots & R_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n-1,m-1}^m R_m(n-1) & \xi_{n-2,m-2}^m R_m(n-2) & \cdots & R_m(n-m) \end{vmatrix} \neq 0$$

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Now consider the sequence of polynomials $(q_n)_n$ defined by

$$q_n(x) = \begin{vmatrix} p_n(x) & -p_{n-1}(x) & \cdots & (-1)^m p_{n-m}(x) \\ \xi^1_{n,m} R_1(n) & \xi^1_{n-1,m-1} R_1(n-1) & \cdots & R_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \xi^m_{n,m} R_m(n) & \xi^m_{n-1,m-1} R_m(n-1) & \cdots & R_m(n-m) \end{vmatrix}$$

Observation: q_n is a linear combination of of $m+1$ consecutive p_n's.

Define for $h = 1, \ldots, m$, the following functions

$$M_h(x) = \sum_{j=1}^{m} (-1)^{h+j} \xi^h_{x,m-j} \det\left(\xi^l_{x+j-r,m-r} R_l(x+j-r)\right)_{\left\{ \begin{array}{c} l \neq h \\ r \neq j \end{array} \right\}}$$

Observation: M_h are linear combinations of adjoint determinants of $\Omega(x)$.

If we assume that $\Omega(x)$ and $M_h(x)$ are polynomials in x, then $\exists\, D_q \in \mathcal{A}$ with $D_q(q_n) = P(n) q_n$ and $P(x) - P(x-1) = \Omega(x)$, where

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Choice of $R_1, R_2, \dots, R_m$

GOAL: Make $(q_n)_n$ bispectral (we already have $D_q(q_n) = \lambda_n q_n$).

For that we have to make an appropriate choice of the arbitrary polynomials $R_1, R_2, \dots, R_m$. This choice is based on the following recurrence formula ($h = 1, \dots, m$):

$$\varepsilon^h_{n+1} a_{n+1} R^h_j(n+1) - b_n R^h_j(n) + \frac{c_n}{\varepsilon^h_n} R^h_j(n-1) = (\eta_h j + \kappa_h) R^h_j(n), \quad n \in \mathbb{Z}$$

where η_h and κ_h are real numbers independent of n and j, $(a_n)_{n\in\mathbb{Z}}$, $(b_n)_{n\in\mathbb{Z}}$, $(c_n)_{n\in\mathbb{Z}}$ are the coefficients in the TTRR for the OP's $(p_n)_n$, and $(\varepsilon^h_n)_n$ defines a $\mathcal{D}$-operator for $(p_n)_n$.

Classical discrete family	$\mathcal{D}$-operators	$R_j(x)$
Charlier: c^a_n, $n \geq 0$	∇	$c^{-a}_j(-x-1)$, $j \geq 0$
Meixner: $m^{a,c}_n$, $n \geq 0$	$\frac{a}{1-a}\Delta$	$m^{1/a,2-c}_j(-x-1)$, $j \geq 0$
	$\frac{1}{1-a}\nabla$	$m^{a,2-c}_j(-x-1)$, $j \geq 0$
Krawtchouk: $k^{a,N}_n$, $n \geq 0$	$\frac{1}{1+a}\nabla$	$k^{a,-N}_j(-x-1)$, $j \geq 0$
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Given a set G of m positive integers, $G = \{g_1, \ldots, g_m\}$, call $\tilde{G} = \{\tilde{g}_1, \ldots, \tilde{g}_m\}$ where $\tilde{g}_h = \eta_h g_h + \kappa_h$.

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$$0 = \sum_{i=1}^m \frac{R^i_{g_i}(n)}{\mathfrak{p}'_{\tilde{G}}(\tilde{g}_i) \xi^i_{-1,-n-1} R^i_{g_i}(-1)}, \quad 1-m \leq n < 0$$

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Identifying the measure $\tilde{\omega}$

$\tilde{\omega}$ will be identified by the Christoffel transform of ω

$$\omega^F(x) = \prod_{f \in F} (x - f)\, \omega(x)$$

The set G will be closely related with the set F.

In fact G will be identified by one of the following sets:

$$I(F) = \{1, 2, \dots, f_k\} \setminus \{f_k - f, f \in F\},$$
$$J_h(F) = \{0, 1, 2, \dots, f_k + h - 1\} \setminus \{f - 1, f \in F\}, \quad h \geq 1$$

where $f_k = \max F$ and $k = \#(F)$.
For the transformation I, the bigger the holes in F (with respect to the set $\{1, 2, \dots, f_k\}$), the bigger the set $I(F)$:

$$I(\{1, 2, 3, \dots, k\}) = \{k\}, \qquad I(\{1, k\}) = \{1, 2, \dots, k-2, k\}$$

IDENTIFYING THE MEASURE $\tilde{\omega}$

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$$I(F) = \{1,2,\dots,f_k\}\setminus\{f_k-f, f\in F\},$$
$$J_h(F) = \{0,1,2,\dots,f_k+h-1\}\setminus\{f-1, f\in F\},\quad h\geq 1$$

where $f_k = \max F$ and $k = \#(F)$.

For the transformation I, the bigger the holes in F (with respect to the set $\{1,2,\dots,f_k\}$), the bigger the set $I(F)$:

$$I(\{1,2,3,\dots,k\}) = \{k\},\qquad I(\{1,k\}) = \{1,2,\dots,k-2,k\}$$

Identifying the measure $\tilde{\omega}$

$\tilde{\omega}$ will be identified by the Christoffel transform of ω

$$\omega^F(x) = \prod_{f \in F} (x - f)\, \omega(x)$$

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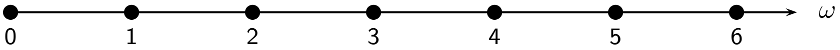
IDENTIFYING THE MEASURE $\tilde{\omega}$: EXAMPLE

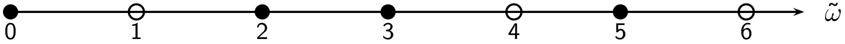
Imagine we have a discrete classical weight ω supported on $\{0, 1, 2, \ldots\}$



Let $F = \{1, 4, 6\}$ and consider the discrete weight ω^F given by

$$\omega^F(x) = \prod_{f \in F}(x - f)\,\omega(x) = (x-1)(x-4)(x-6)\,\omega(x)$$

The new discrete weight ω^F will be supported on $\{0, 2, 3, 5, 7 \ldots\}$



The set of indexes G we have to take to construct the orthogonal polynomials $(q_n^G)_n$ with respect to $\tilde{\omega} = \omega^F$ will be given by

$$G = I(F) = \{1, 2, 3, 4, 5, 6\} \setminus \{5, 2, 0\} = \{1, 3, 4, 6\}$$

Identifying the measure $\tilde{\omega}$: example

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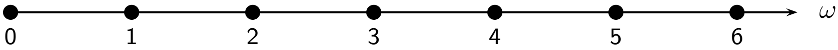

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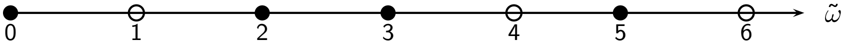

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Outline

1 Introduction
- Classical orthogonal polynomials
- Krall orthogonal polynomials

2 Methodology
- $\mathcal{D}$-operators
- Choice of arbitrary polynomials
- Identifying the measure

3 Examples
- Charlier, Meixner and Krawtchouk polynomials
- Laguerre polynomials

Charlier polynomials

Let $F \subset \mathbb{N}$ be finite and consider $G = I(F) = \{g_1, \dots, g_m\}$.
Let ω_a be the Charlier measure and $(c_n^a)_n$ its sequence of OP's. Assume that $\Omega_G(n) = \det\left(c_{g_l}^{-a}(-n-j-1)\right)_{l,j=1}^{m} \neq 0$.

If we define $(q_n)_n$ by

$$q_n(x) = \begin{vmatrix} c_n^a(x) & -c_{n-1}^a(x) & \cdots & (-1)^m c_{n-m}^a(x) \\ c_{g_1}^{-a}(-n-1) & c_{g_1}^{-a}(-n) & \cdots & c_{g_1}^{-a}(-n+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ c_{g_m}^{-a}(-n-1) & c_{g_m}^{-a}(-n) & \cdots & c_{g_m}^{-a}(-n+m-1) \end{vmatrix}$$

then the polynomials $(q_n)_n$ are orthogonal with respect to the measure

$$\omega_a^F = \prod_{f \in F} (x - f)\omega_a$$

and they are eigenfunctions of a higher order difference operator D_q with

$$-s = r = \sum_{f \in F} f - \frac{k(k-1)}{2} + 1, \quad k = \#(F)$$

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Meixner polynomials

In this case have two different $\mathcal{D}$-operators. That means that we will have to consider two sets of positive integers $F_1, F_2 \subset \mathbb{N}$.

Consider $H = J_h(F_1) = \{h_1, \ldots, h_{m_1}\}$ and $K = I(F_2) = \{k_1, \ldots, k_{m_2}\}$.

Define $m = m_1 + m_2$ and consider the Meixner polynomials $(m_n^{a,c})_n$. Assume that

$$\Omega_{a,c}^{H,K}(n) = \begin{vmatrix} m_{h_1}^{1/a,2-c}(-n) & \cdots & m_{h_1}^{1/a,2-c}(-n+m-1) \\ \vdots & \ddots & \vdots \\ m_{h_{m_1}}^{1/a,2-c}(-n) & \cdots & m_{h_{m_1}}^{1/a,2-c}(-n+m-1) \\ \dfrac{m_{k_1}^{a,2-c}(-n)}{a^{m-1}} & \cdots & m_{k_1}^{a,2-c}(-n+m-1) \\ \vdots & \ddots & \vdots \\ \dfrac{m_{k_{m_2}}^{a,2-c}(-n)}{a^{m-1}} & \cdots & m_{k_{m_2}}^{a,2-c}(-n+m-1) \end{vmatrix} \neq 0$$

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Meixner polynomials

If we define $(q_n)_n$ by

$$q_n(x) = \begin{vmatrix} \dfrac{(1-a)^m m_n^{a,c}(x)}{a^m} & \dfrac{-(1-a)^{m-1} m_{n-1}^{a,c}(x)}{a^{m-1}} & \cdots & (-1)^m m_{n-m}^{a,c}(x) \\ m_{h_1}^{1/a,2-c}(-n-1) & m_{h_1}^{1/a,2-c}(-n) & \cdots & m_{h_1}^{1/a,2-c}(-n+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ m_{h_{m_1}}^{1/a,2-c}(-n-1) & m_{h_{m_1}}^{1/a,2-c}(-n) & \cdots & m_{h_{m_1}}^{1/a,2-c}(-n+m-1) \\ \dfrac{m_{k_1}^{a,2-c}(-n-1)}{a^m} & \dfrac{m_{k_1}^{a,2-c}(-n)}{a^{m-1}} & \cdots & m_{k_1}^{a,2-c}(-n+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{m_{k_{m_2}}^{a,2-c}(-n-1)}{a^m} & \dfrac{m_{k_{m_2}}^{a,2-c}(-n)}{a^{m-1}} & \cdots & m_{k_{m_2}}^{a,2-c}(-n+m-1) \end{vmatrix}$$

then the polynomials $(q_n)_n$ are eigenfunctions of a higher order difference operator D_q and they are orthogonal with respect to the measure

$$\omega_{a,c}^{F_1,F_2} = \prod_{f\in F_1}(x+c+f)\prod_{f\in F_2}(x-f)\omega_{a,c}$$

Krawtchouk polynomials

Again, for $F_1, F_2 \subset \mathbb{N}$ consider $K = I(F_1) = \{k_1, \dots, k_{m_2}\}$ and $H = J_h(F_2) = \{h_1, \dots, h_{m_1}\}$ with $m = m_1 + m_2$.

If we define $(q_n)_n$ by

$$q_n(x) = \begin{vmatrix} (1+a)^m k_n^{a,N}(x) & -(1+a)^{m-1} k_{n-1}^{a,N}(x) & \cdots & (-1)^m k_{n-m}^{a,N}(x) \\ k_{k_1}^{a,-N}(-n-1) & k_{k_1}^{a,-N}(-n) & \cdots & k_{k_1}^{a,-N}(-n+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ k_{k_{m_1}}^{a,-N}(-n-1) & k_{k_{m_1}}^{a,-N}(-n) & \cdots & k_{k_{m_1}}^{a,-N}(-n+m-1) \\ (-a)^m k_{h_1}^{1/a,-N}(-n-1) & (-a)^{m-1} k_{h_1}^{1/a,-N}(-n) & \cdots & k_{h_1}^{1/a,-N}(-n+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ (-a)^m k_{h_{m_2}}^{1/a,-N}(-n-1) & (-a)^{m-1} k_{h_{m_2}}^{1/a,-N}(-n) & \cdots & k_{h_{m_2}}^{1/a,-N}(-n+m-1) \end{vmatrix}$$

then the polynomials $(q_n)_n$ are eigenfunctions of a higher order difference operator D_q and orthogonal with respect to the measure

$$\omega_{a,N}^{F_1,F_2} = \prod_{f \in F_1} (x-f) \prod_{f \in F_2} (N-1-f-x) \omega_{a,N}$$

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Laguerre polynomials

For $m \geq 1$, let $M = (M_{i,j})_{i,j=0}^{m-1}$ be any $m \times m$ matrix. For $\alpha \neq m-1, m-2, \ldots$, consider the discrete Laguerre-Sobolev bilinear form defined by

$$\langle p, q \rangle = \int_0^\infty p(x)q(x)x^{\alpha-m}e^{-x}dx + (p(0), \ldots, p^{(m-1)}(0))M \begin{pmatrix} q(0) \\ \vdots \\ q^{(m-1)}(0) \end{pmatrix}$$

Then the family $(q_n)_n$ defined by

$$q_n(x) = \begin{vmatrix} L_n^\alpha(x) & L_{n-1}^\alpha(x) & \cdots & L_{n-m}^\alpha(x) \\ \mathcal{R}_1(n) & \mathcal{R}_1(n-1) & \cdots & \mathcal{R}_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{R}_m(n) & \mathcal{R}_m(n-1) & \cdots & \mathcal{R}_m(n-m) \end{vmatrix}, \quad n \geq 0$$

is orthogonal with respect to the discrete Laguerre-Sobolev bilinear form, as long as $\Omega(n) = \det(\mathcal{R}_i(n-j))_{i,j=1}^m \neq 0, n \geq 0$, where

$$\mathcal{R}_l(x) = \frac{\Gamma(\alpha - m + l)}{(m-l)!}(x+1)_{m-l} + (l-1)!\frac{\Gamma(\alpha+1+x)}{\Gamma(1+x)}\sum_{i=0}^{m-1}\frac{(-1)^i M_{l-1,i}}{\Gamma(\alpha+i+1)}(x-i+1)_i$$

Observation: $\mathcal{R}_1(x), \ldots, \mathcal{R}_m(x)$ are not polynomials in general.

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Laguerre polynomials

Let $(L_n^\alpha)_n$ be the family of Laguerre polynomials and D_p the corresponding second-order differential equation such that $D_p(L_n^\alpha) = nL_n^\alpha$.
Assume that α is a positive integer with $\alpha \geq m$.
Then there exists a differential operator D_q of the form

$$D_q = P(D_p) + \sum_{h=1}^{m} M_h(D_p)\frac{d}{dx}\mathcal{R}_h(D_p),$$

such that $D_q(q_n) = P(n)q_n$ where

$$P(x) - P(x-1) = \Omega(x)$$

and the polynomials $M_h(x), h = 1, \dots, m$ are defined by

$$M_h(x) = \sum_{j=1}^{m}(-1)^{h+j}\det\left(\mathcal{R}_l(x+j-r)\right)_{\left\{\begin{array}{c} l \neq h \\ r \neq j \end{array}\right\}}$$

$$\mathcal{R}_l(x) = \frac{(\alpha-m+l-1)!}{(m-l)!}(x+1)_{m-l}+(l-1)!(x+1)_\alpha\sum_{i=0}^{m-1}\frac{(-1)^i M_{l-1,i}}{(\alpha+i)!}(x-i+1)_i$$

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Laguerre polynomials

Moreover, the minimal order of the differential operator D_q having the orthogonal polynomials $(q_n)_n$ as eigenfunctions is at most $2(\alpha\text{-wr}(M)+1)$ where $\alpha\text{-wr}(M)$ is the α-weighted rank of the matrix M, given by

$$\alpha\text{-wr}(M) = \sum_{j=1}^{m} n_j + \sum_{j=1}^{m-1} m_j - \frac{m(m-1)}{2}$$

The indexes n_j and m_j are related with how singular are the columns and the rows of the matrix M.

- When $M = (M_{i,j})_{i,j=0}^{m-1}$ is the symmetric matrix with entries $M_{i,j} = a_{i+j}$ for $i+j \leq m-1$ and $M_{i,j} = 0$ for $i+j > m-1$, the discrete Laguerre Sobolev inner product reduces

$$x^{\alpha-m}e^{-x} + \sum_{i=0}^{m-1} a_i \delta_0^{(i)}, \quad \alpha\text{-wr}(M) = m\alpha$$

- When M is diagonal, $M = \operatorname{diag}(M_0, \dots, M_{m-1})$, $M_{m-1} \neq 0$, we have

$$\alpha\text{-wr}(M) = s\alpha + (m-s)(m+1) - 2 \sum_{1\leq j\leq m, M_{j-1}=0} j, \quad s = |\{j : 1 \leq j \leq m, M_j \neq 0\}|$$

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Laguerre polynomials: explicit example

Let $\alpha = 3$, $m = 3$ and $M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $\mathcal{R}_1(x), \mathcal{R}_2(x), \mathcal{R}_3(x)$ are given by

$$\begin{aligned} \mathcal{R}_1(x) &= -\frac{(x+1)(x+2)(x^2-x-24)}{24} \\ \mathcal{R}_2(x) &= -\frac{(x+1)(x^3+x^2-14x-48)}{24} \\ \mathcal{R}_3(x) &= \frac{(x+4)(x^4+x^3+x^2-9x+30)}{60} \end{aligned}$$

The differential operator (of order 18) satisfying $D_q(q_n) = P(n)q_n$ is

$$D_q = P(D_p) + \sum_{h=1}^{3} M_h(D_p)\frac{d}{dx}\mathcal{R}_h(D_p)$$

where

$$P(x) = -\frac{x^9}{4320} + \frac{x^8}{480} - \frac{x^7}{144} - \frac{17x^6}{720} + \frac{47x^5}{480} - \frac{253x^4}{1440} + \frac{55x^3}{108} - \frac{289x^2}{360} - \frac{18x}{5}$$

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$$P(x) = -\frac{x^9}{4320} + \frac{x^8}{480} - \frac{x^7}{144} - \frac{17x^6}{720} + \frac{47x^5}{480} - \frac{253x^4}{1440} + \frac{55x^3}{108} - \frac{289x^2}{360} - \frac{18x}{5}$$

Laguerre polynomials: explicit example

Let $\alpha = 3$, $m = 3$ and $M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $\mathcal{R}_1(x), \mathcal{R}_2(x), \mathcal{R}_3(x)$ are given by

$$\mathcal{R}_1(x) = -\frac{(x+1)(x+2)(x^2-x-24)}{24}$$

$$\mathcal{R}_2(x) = -\frac{(x+1)(x^3+x^2-14x-48)}{24}$$

$$\mathcal{R}_3(x) = \frac{(x+4)(x^4+x^3+x^2-9x+30)}{60}$$

The differential operator (of order 18) satisfying $D_q(q_n) = P(n)q_n$ is

$$D_q = P(D_p) + \sum_{h=1}^{3} M_h(D_p)\frac{d}{dx}\mathcal{R}_h(D_p)$$

where

$$P(x) = -\frac{x^9}{4320} + \frac{x^8}{480} - \frac{x^7}{144} - \frac{17x^6}{720} + \frac{47x^5}{480} - \frac{253x^4}{1440} + \frac{55x^3}{108} - \frac{289x^2}{360} - \frac{18x}{5}$$

Conclusions and extensions

Conclusions: Given a classical family of OP's $(p_n)_n$ with a second-order difference or differential operator D_p such that $D_p(p_n) = np_n$ (Charlier, Meixner, Krawtchouk and Laguerre) we can construct a new bispectral family of OP's satisfying higher-order difference or differential operators.

A. J. Durán and M. D. de la Iglesia, *Constructing bispectral orthogonal polynomials from the classical discrete families of Charlier, Meixner and Krawtchouk*, to appear in Constructive Approximation.

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Future work: Examples of the form D_p with $D_p(p_n) = \theta_n p_n$, where θ_n is any function of n. The classical families to study in this case are the Jacobi (continuous) and Hahn (discrete).

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